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EMBEDDING GROUPS INTO DISTRIBUTIVE SUBSETS OF THE MONOID OF BINARY OPERATIONS

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ABSTRACT. Let X be a set and $Bin(X)$ the set of all binary operations on X. We say that $S \subset Bin(X)$ is a distributive set of operations if all pairs of elements $*_\alpha, *_\beta \in S$ are right distributive, that is, $(a *_\alpha b) *_\beta c = (a *_\beta c) *_\alpha (b *_\beta c)$ (we allow $*_\alpha = *_\beta$).

J.Przytycki posed the question of which groups can be realized as distributive sets. The initial guess that any group may be embedded into $Bin(X)$ for some X was complicated by an observation that if $* \in S$ is idempotent ($a*a = a$), then $*$ commutes with every element of S. The first noncommutative subgroup of $Bin(X)$ (the group S_3) was found in October of 2011 by Y.Berman.

We show that any group can be embedded in $Bin(X)$ for $X = G$ (as a set). We also discuss minimality of embeddings observing that, in particular, X with six elements is the smallest set such that $Bin(X)$ contains a non-abelian subgroup.

Contents

1. Introduction

Let X be a set and $Bin(X)$ the set of all binary operations on X. We say that $S \subset Bin(X)$ is a distributive set of operations if all pairs of elements $*_\alpha, *_\beta \in S$ are right distributive, that is, $(a *_\alpha b) *_\beta c =$

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$(a *_\beta c) *_\alpha (b *_\beta c)$ (we allow $*_\alpha = *_\beta$). $(X; S)$ is called a multi-shelf[1] in this case. It was observed in [Prz-1] (compare also [Ro-S]) that $Bin(X)$ is a monoid with composition $*_1 *_2$ given by $a *_1 *_2 b = (a *_1 b) *_2 b$, with the identity $*_0$ being the right trivial operation, that is, $a *_0 b = a$ for any $a, b \in X$.

The submonoid of $Bin(X)$ of all invertible elements in $Bin(X)$ is a group denoted by $Bin_{inv}(X)$. If $* \in Bin_{inv}(X)$ then $*^{-1}$ is usually denoted by $\bar{*}$.

We say that a subset $S \subset Bin(X)$ is a distributive set if all pairs of elements $*_\alpha, *_\beta \in S$ are right distributive, that is, $(a *_\alpha b) *_\beta c = (a *_\beta c) *_\alpha (b *_\beta c)$ (we allow $*_\alpha = *_\beta$).

The following important basic lemma was proven in [Prz-1]:

Lemma 1.1.
(i) *If S is a distributive set and $* \in S$ is invertible, then $S \cup \{\bar{*}\}$ is also a distributive set.*
(ii) *If S is a distributive set and $M(S)$ is the monoid generated by S, then $M(S)$ is a distributive monoid.*
(iii) *If S is a distributive set of invertible operations and $G(S)$ is the group generated by S, then $G(S)$ is a distributive group.*

The question was asked by J.Przytycki which groups can be realized as distributive sets. Soon after the definition of a distributive submonoid of $Bin(X)$ was given in [Prz-1], Michal Jablonowski, a graduate student at Gdańsk University, noticed that any distributive monoid whose elements are idempotent operations is commutative. We have:

Proposition 1.2. *(*[Prz-1]*)*
(i) *Consider $*_\alpha, *_\beta \in Bin(X)$ such that $*_\beta$ is idempotent ($a *_\beta a = a$) and distributive with respect to $*_\alpha$. Then $*_\alpha$ and $*_\beta$ commute. In particular:*
(ii) *If M is a distributive monoid and $*_\beta \in M$ is an idempotent operation, then $*_\beta$ is in the center of M.*
(iii) *A distributive monoid whose elements are idempotent operations is commutative.*

Proof. We have: $(a *_\alpha b) *_\beta b \overset{distrib}{=} (a *_\beta b) *_\alpha (b *_\beta b) \overset{idemp}{=} (a *_\beta b) *_\alpha b$. $\square$

A few months later Agata Jastrzębska (also a graduate student at Gdańsk University) checked that any distributive group in $Bin_{inv}(X)$ for $|X| \le 5$ is commutative.

The first noncommutative subgroup of $Bin(X)$ (the group S_3) was found in October of 2011 by Yosef Berman. Soon after Berman (with

[1] If $(X; *)$ is a magma and $*$ is a right self-distributive operation, then $(X; *)$ is called a shelf - the term coined by Alissa Crans in her PhD thesis [Cr].

the help of Carl Hammarsten) constructed an embedding of a general dihedral group $D_{2\cdot n}$ in $Bin(X)$ where X has $2n$ elements. The embedding of Berman $\phi : D_{2\cdot 3} \to Bin(X)$ is given as follows: if $X = \{0,1,2,3,4,5\}$ then the subgroup $D_{2\cdot 3} \subset Bin(X)$ is generated by the binary operations $*_\tau$ (reflection) and $*_\sigma$ (a 3-cycle):

$$*_\tau = \begin{pmatrix} 1 & 1 & 3 & 5 & 5 & 3 \\ 0 & 0 & 4 & 2 & 2 & 4 \\ 3 & 3 & 5 & 1 & 1 & 5 \\ 2 & 2 & 0 & 4 & 4 & 0 \\ 5 & 5 & 1 & 3 & 3 & 1 \\ 4 & 4 & 2 & 0 & 0 & 2 \end{pmatrix} \text{ and } *_\sigma = \begin{pmatrix} 2 & 4 & 2 & 4 & 2 & 4 \\ 5 & 3 & 5 & 3 & 5 & 3 \\ 4 & 0 & 4 & 0 & 4 & 0 \\ 1 & 5 & 1 & 5 & 1 & 5 \\ 0 & 2 & 0 & 2 & 0 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 \end{pmatrix}.$$

where $i * j$ is placed in the ith row and jth column, and $D_{2\cdot 3} = \{\tau, \sigma \mid \tau\sigma\tau = \sigma^{-1}\}$.

2. Regular distributive embedding

We now show that any group G can be embedded in $Bin(X)$ for some X.

Theorem 2.1. *(Regular embedding)*
Every group G embeds in $Bin(G)$. This embedding (monomorphism), $\phi^{reg} : G \to Bin(G)$ sends g to $*_g$ where $a *_g b = ab^{-1}gb$.

Proof. (i) We check that the set $\{*_g\}_{g\in G}$ is a distributive set. We have: $(a *_{g_1} b) *_{g_2} c = (ab^{-1}g_1b) *_{g_2} c = ab^{-1}g_1bc^{-1}g_2c$, and

$$(a*_{g_2}c)*_{g_1}(b*_{g_2}c) = (ac^{-1}g_2c)*_{g_1}(bc^{-1}g_2c) = ab^{-1}g_1bc^{-1}g_2c, \text{ as needed.}$$

(ii) Now we check that the map ϕ^{reg} is a monomorphism. Of course the image of the identity $*_0$ is the identity in $Bin(G)$. Furthermore: $a *_{g_1g_2} b = ab^{-1}g_1g_2b$, and
$a *_{g_1} *_{g_2} b = (a *_{g_1} b) *_{g_2} b = ab^{-1}g_1bb^{-1}g_2b = ab^{-1}g_1g_2b$, as needed. We have proven that ϕ^{reg} is a homomorphism. To show that ϕ^{reg} is a monomorphism we substitute $b = 1$ in the formula for $a *_g b$, to get $a *_g 1 = ag$, so different g's give different binary operations in $Bin(G)$. Notice that $\phi^{reg}(g^{-1}) = \bar{*}_g$. □

We call our embedding *regular* by analogy to the regular representation of a group. We do not claim that the regular embedding is minimal. In fact, finding minimal distributive embeddings is a very interesting problem in itself.

3. General conditions for a distributive embedding

We now discuss a method that can be used to embed groups into subsets of $Bin_{inv}(X)$ satisfying a given condition. We then use this method when the condition is right distributivity, which led us to discover the regular distributive embedding of G in $Bin(G)$, and also should be a natural tool to look for minimal embeddings. For the group S_3 we know, by Jastrzebska's calculations, that X with six elements is the minimal set such that S_3 embeds in $Bin(X)$.

We start from the following basic observation:

Lemma 3.1. *There is an isomorphism between $Bin_{inv}(X)$ and $S_X^{|X|}$, where $|X|$ is the cardinality of $|X|$ and S_X is the group of permutation on set X (i.e. bijections of the set X). The isomorphism α : $Bin_{inv}(X) \to S_X^{|X|} = \Pi_{y\in X} S_X^y$ is described as follows: $\alpha(*)(y) : X \to X$ is the bijection where $(\alpha(*)(y))(x) = x * y$. In other words $\alpha(*)(y)$ is the bijection corresponding to the y coordinate of $S_X^{|X|}$.*

Using the map α, we can translate conditions on a set of binary operations in $Bin(X)$ into a group-theoretic condition on (coordinates of) elements of $S_X^{|X|}$. With some work, we can use this to find an embedding of a group into $Bin(X)$. This is possible since the group axioms require that such an embedding sits inside $Bin_{inv}(X)$. Let us consider distributive, invertible sets $\mathcal{S}$ of binary operations in $Bin_{inv}(X)$. These are subsets $\mathcal{S} \subseteq Bin_{inv}(X)$ that satisfy:

$$(x *_i y) *_j z = (x *_j z) *_i (y *_j z), \ for\ all\ *_i, *_j \in S\ and\ x, y, z \in X.$$

Let $\sigma_{i,y} = p_y\alpha(*_i)$, where $p_y : S_X^{|X|} \to S_X$ is projection onto the y^{th} coordinate. Then translating the distributivity condition via α:

$$\sigma_{j,z}(x *_i y) = \sigma_{i,(y*_j z)}(x *_j z),$$

or

$$\sigma_{j,z}(\sigma_{i,y}(x)) = \sigma_{i,\sigma_{j,z}(y)}(\sigma_{j,z}(x)),$$

which leads to

$$\sigma_{i,\sigma_{j,z}(y)} = \sigma_{j,z}\sigma_{i,y}\sigma_{j,z}^{-1}.$$

Now the problem of embedding a group into $Bin_{inv}(X)$ is reduced to finding subsets of $S_X^{|X|}$ isomorphic to the group that satisfy the condition above. We can then use tools of group theory (e.g., representation theory) to solve the problem. This process can by attempted for subsets of $Bin_{inv}(X)$ satisfying any condition, and led to the embedding defined in the previous section for distributive subsets.

4. Future directions; multi-term homology

Przytycki defined multi-term homology for any distributive set in [Prz-1]. This provided motivation to have examples of distributive sets. The regular embedding of a group (Theorem 2.1) provides an interesting family of distributive sets ripe for studying this homology (compare [CPP, Prz-1, Prz-2, Pr-Pu, P-S]). As a nontrivial example we propose computing n-term distributive homology related with the regular embedding of the cyclic group Z_n. Another problem related to Theorem 2.1 is which monoids are distributive submonoids of $Bin(X)$.

A key motivation is to use multi-term distributive homology in knot theory. This possibility arises from the relation of the third Reidemeister move with right distributivity (and eventually the Yang-Baxter operator), and the important work of Carter, Kamada, Saito, and other researchers on applications of quandle homology to knot theory (see [CKS]).

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